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Asymptotics of Eigenvalues for Sturm–Liouville Problems with an Interior Singularity

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We consider the asymptotic form of the eigenvalues of the linear differential equation

$$-y''(x)+q(x)\,y(x)=\lambda y(x), \qquad -\infty<a<x<b<\infty,$$

where $a<0<b$, $q(x)$ is singular at $x=0$, and y satisfies appropriate conditions at a, 0, and b. This extends previous work of Atkinson and of Harris. In particular, when $q(x)=x^{-K}$, Atkinson derived asymptotic formulae which cover the case $1\leqslant K<\frac{4}{3}$; Harris's results cover the cases $1\leqslant K<\frac{3}{2}$. We now cover all of the cases $1\leqslant K<2$. Since the methods employed by both of these authors and ourselves apply only to limit circle, non-oscillatory expressions, our results now seem to take problems of this type to their conclusion.

1. BACKGROUND

In recent years there has been increasing interest in spectral problems associated with linear differential equations of the form

$$-y''(x)+q(x)\,y(x)=\lambda y(x), \qquad -\infty<a<x<b<\infty, \tag{1.1}$$

in which λ is a real parameter and q is a real-valued function which has a singularity at some point $c\in(a,b)$. That is, q is not Lebesgue-integrable at c. Typically, $c=0$ in the examples. The Boyd equation in which $q(x)=-x^{-1}$ is probably the best known and most investigated example of this behaviour (see [3, 4, 7, 8, and the references cited therein.) Some of these investigations considered purely the Boyd equation while others have

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examined broad classes of coefficients q. However, the context is of those q for which the expression on the left of (1.1) is limit circle and non-oscillatory; i.e., all solutions of (1.1) lie in $L^2(a, b)$ for one (and hence every) λ and possess only a finite number of zeros in $[a, b]$. For $q(x) = Cx^{-K}$ these properties hold when $1 \leqslant K < 2$.

Atkinson and Fulton found asymptotic formulae for the eigenvalues of such problems when considered on the interval $(0, b]$, it being known that they are bounded below and satisfy $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots, \lambda_n \to \infty$ as $n \to \infty$. The case $q(x) = Cx^{-1}$ was examined there in greater detail. In subsequent work, Atkinson [3] considered the more intricate question of finding such eigenvalue asymptotics for an interior (rather than an endpoint) singularity. His results cover the case $q(x) = Cx^{-K}$ for $1 \leqslant K < \frac{4}{3}$. In a refinement of Atkinson's work, Harris [10] achieved analogous formulae for a class of q which covers the above examples for $1 \leqslant K < \frac{3}{2}$.

The aim of the present work is to find similar eigenvalue asymptotics for potentials q which are more singular than previously covered. In particular we cover the full range $1 \leqslant K < 2$ in the example $q(x) = Cx^{-K}$. To achieve this we have to introduce several new ideas and approximation techniques into the approach of [2, 3, 10].

By means of suitable transformations (see, for example, [6]), equations of the form

$$-(py')' + qy = \lambda wy \tag{1.2}$$

may be transformed into the form (1.1) such that many properties are preserved. Since the asymptotic formulae obtained for (1.1) are already quite involved, we confine ourselves to (1.1) here and merely observe that our results may be applied to certain equations of the form (1.2) having an interior singularity by means of such transformations.

Our techniques include rewriting the singular equations (1.1) as a quasi-differential equation which is regular at the original singular point. Since such a regularization is possible if and only if the expression on the left of (1.1) is both limit circle and non-oscillatory (see [9] cf. [8, 11]), it appears that our results extend these techniques as far as possible, at least for examples such as $q(x) = Cx^{-K}$.

2. Regularization

Initially we follow the approach of [3, 10]. We suppose that in (1.1),

$$q \in L_{\mathrm{loc}}[a, 0) \oplus L_{\mathrm{loc}}(0, b] \tag{2.1}$$

assuming without loss of generality that q has a singularity at 0 (i.e., taking $c=0$). Thus, we assume

$$q \notin L[a, b]. \tag{2.2}$$

We then suppose that there exists some real-valued function f on $[a, 0) \cup (0, b]$ in $AC_{\text{loc}}([a, 0) \cup (0, b])$ which regularizes (1.1) in the following sense. Define the quasi-derivative $y^{[1]}$ of a function y by

$$y^{[1]} := y' + fy. \tag{2.3}$$

Then y is a solution of (1.1) if and only if

$$(y^{[1]})' - fy^{[1]} + Fy = 0, \tag{2.4}$$

where

$$F = -q - f' + f^2. \tag{2.5}$$

Thus (1.1) is equivalent to the first order system

$$\begin{pmatrix} y \\ y^{[1]} \end{pmatrix}' = \begin{pmatrix} -f & 1 \\ -(F+\lambda) & f \end{pmatrix} \begin{pmatrix} y \\ y^{[1]} \end{pmatrix}. \tag{2.6}$$

We assume that f regularizes (1.1) in the sense that the system (2.6) is regular; that is

$$f, F \in L(a, b). \tag{2.7}$$

Since we are using Lebesgue measure, we need not distinguish between functions defined on $[a, 0) \cup (0, b]$ or on $[a, b]$, as regards their integrability. For this type of regularization using quasi-derivatives see [3, Section 3, 8, 9].

In [3] the special choice of $f = -\int^x q$ was made, and it was necessary to assume that $f^3 \in L(a, b)$. In [10] a more sophisticated choice of f was made, and a set of properties including $f^2, fF \in L(a, b)$ had to be assumed. Appropriate asymptotic formulae for the eigenvalues of (1.1) could then be derived. Here, our aim is to assume little more than (2.7) since an appropriate choice of f can then be made to regularize (1.1) when

$$q(x) = Cx^{-K} \tag{2.8}$$

for any K in the range $1 \leqslant K < 2$ (see [8, Example 5]).

We observe that if $f = f_0$ and $F = F_0$ provides a regularization of (1.1) through (2.3)–(2.7), then so does $f = f_0 + c_0$ and $F = F_0 + 2c_0 f_0 + c_0^2$ for any real constant c_0 (see [3, 8]). Furthermore, we could choose one value of c_0 in $(a, 0)$ and another value on $(0, b)$.

In [3] it was pointed out that in using the regularization in (2.3)–(2.7) one is thereby specifiying two particular interface conditions across $x=0$, namely that solutions of (1.1) satisfy

$$\begin{aligned} \lim_{x\to 0^-} y(x) &= \lim_{x\to 0^+} y(x) \\ \lim_{x\to 0^-} (y'(x)+f(x)\,y(x)) &= \lim_{x\to 0^+} (y'(x)+f(x)\,y(x)). \end{aligned} \tag{2.9}$$

These correspond to requiring the continuity of y and of $y^{[1]}$ across $x=0$. These same interface conditions were present in [10]. The effect on the asymptotic formulae of introducing different constants c_0 as above on either side of $x=0$ was discussed by Atkinson in [3, Section 4]. We reiterate that this choice is expected to affect the estimates for the eigenvalues only in the higher order terms.

3. The Modified Prüfer Transformation

Having rewritten Eq. (1.1) as the system (2.6), we now follow [3, 10] in observing that, for any solution y of (1.1) with $\lambda>0$, we may define a function $\theta\in AC[a,b]$ by

$$\tan\theta=\lambda^{1/2}y/y^{[1]}. \tag{3.1}$$

When $y^{[1]}=0$, θ is defined by continuity. The regularity of the system (2.6) ensures that there is no difficulty at $x=0$. The function θ is determined uniquely to within additive multiples of π by (3.1) and from (2.6) satisfies the differential equation

$$\theta'=\lambda^{1/2}-f\sin(2\theta)+\lambda^{-1/2}F\sin^2\theta. \tag{3.2}$$

An argument along the lines of [5, pp. 208–213] (see [2, Theorem 2] and [3, Section 5]) shows that for Dirichlet boundary conditions at a and b, the positive eigenvalues λ_n of (1.1) subject to (2.9) at $x=0$ are determined by

$$\theta(a,\lambda_n)=0, \qquad \theta(b,\lambda_n)=(n+1)\,\pi \tag{3.3}$$

for n a sufficiently large natural number. For the corresponding problem with Neumann boundary conditions at a and b (in the sense of $y^{[1]}=0$) the positive eigenvalues are determined for large n by

$$\theta(a,\lambda_n)=\frac{\pi}{2}, \qquad \theta(b,\lambda_n)=\left(n+\frac{1}{2}\right)\pi. \tag{3.4}$$

For a discussion of other boundary conditions at a and b see [2, Section 3], where the situation for a singular endpoint is fully discussed.

It follows easily from (2.7), (3.2)–(3.4) that large positive eigenvalues of either the Dirichlet or Neumann problems over $[a, b]$ satisfy

$$\lambda_n^{1/2} = (n+1)\,\pi/(b-a) + O(1), \tag{3.5}$$

which after reversion (see [12, pp. 21, 22]) leads to an error of $O(n)$ in the estimate of λ_n. Our aim here is to obtain formulae like (3.5) in which the $O(1)$ term is replaced by an integral term which can be calculated explicitly in examples, plus an error term of smaller order. We obtain an error term of $o(\lambda^{-1/2})$ which leads to an error of $o(n^{-1})$ in $\lambda_n^{1/2}$ and hence $o(1)$ in λ_n. To achieve this we first use the differential equation (3.2) to obtain estimates for $\theta(b)$-$\theta(a)$ for general λ as $\lambda \to \infty$.

4. The Main Estimate for θ

In obtaining results under weaker assumptions than those in [3, 10] there is a price to be paid. Namely, we have to use an iterative procedure even for our first estimate for θ, then introduce a further iterative procedure for approximating certain of the terms which appear. This all requires a number of technical lemmas whose proofs we delay to later sections. Having completed this process, however, we can demonstrate the application of these estimates to explicit examples. In this section we state our two main estimates for θ, stating all the necessary definitions, but implicitly making the assumptions of Sections 2 and 3 above.

For a natural number N to be chosen later, we define a sequence $\{\phi_j(t, \lambda)\}$ for $j = 0, 1, ..., N$, $\lambda > 0$, $t \in [a, b]$ by

$$\begin{aligned} \phi_0(t) &:= \theta(0) + \lambda^{1/2} t \\ \phi_j(0) &:= \theta(0) \\ \phi_j'(t) &:= \lambda^{1/2} - f(t)\sin(2\phi_{j-1}(t)) + \lambda^{-1/2} F(t)\sin^2(\phi_{j-1}(t)). \end{aligned} \tag{4.1}$$

We also define a sequence $\{\xi_j(t)\}$ for $j = 1, ..., N+1$ $t \in [a, b]$ by

$$\begin{aligned} \xi_1(t) &:= \left| \int_0^t |f(s)| + |F(s)|\, ds \right| \\ \xi_j(t) &:= \left| \int_0^t (|f(s)| + |F(s)|)\, \xi_{j-1}(s)\, ds \right| \end{aligned} \tag{4.2}$$

and note that in view of (2.7),

$$\xi_j(t) \leqslant C\xi_{j-1}(t) \qquad \text{for} \quad t \in [a, b], \qquad 2 \leqslant j \leqslant N+1. \tag{4.3}$$

Here, and throughout, we use C generically to denote a suitable constant.

THEOREM 4.1. *Suppose that for some* $N \geqslant 1$,

$$\begin{gathered} f'\xi_{N+1}, \qquad f^2\xi_N, \qquad fF\xi_N \in L[a,b]; \\ f(t)\,\xi_{N+1}(t) \to 0 \qquad \text{as} \quad t \to 0. \end{gathered} \tag{4.4}$$

Then as $\lambda \to \infty$,

$$\begin{aligned} \theta(b)-\theta(a) = \lambda^{1/2}(b-a) - \int_a^b f(t)\sin(2\phi_N(t))\,dt + \tfrac{1}{2}\lambda^{-1/2}\int_a^b F(t)\,dt \\ - \lambda^{-1/2}[f(t)(\sin^2\theta(t)-\sin^2\phi_N(t))]_a^b + o(\lambda^{-1/2}). \end{aligned} \tag{4.5}$$

COROLLARY 4.2. *Under the assumptions* (4.4),

$$\begin{aligned} \theta(0)-\theta(a) = -\lambda^{1/2}a - \int_a^0 f(t)\sin(2\phi_N(t))\,dt + \tfrac{1}{2}\lambda^{-1/2}\int_a^0 F(t)\,dt \\ + \lambda^{-1/2}f(a)(\sin^2\theta(a)-\sin^2\phi_N(a)) + o(\lambda^{-1/2}) \end{aligned} \tag{4.6}$$

as $\lambda \to \infty$.

We give the proof of these results in Section 5 below, along with three lemmas which are needed en route. The following example demonstrates how the conditions (4.4) may hold even when those needed in [3, 10] do not, so that Theorem 4.1 covers situations not covered by [3, Theorem 1] or [10, Theorem 3.1], which are its predecessors.

EXAMPLE 4.3. Suppose $|f(t)| \leqslant C\,|t|^{-1+\varepsilon}$ and $|F(t)| \leqslant C\,|t|^{-1+\varepsilon}$ for some $\varepsilon > 0$ on $[a,b]$. Then $|\xi_j(t)| \leqslant C\,|t|^{j\varepsilon}$ for $t \in [a,b]$, $1 \leqslant j \leqslant N+1$.

The details follow immediately from (4.2) by induction on j. Furthermore, it was shown in [8, Example 5] how f may be constructed so that these conditions are satisfied when $q(x) = k\,|x|^{-2+\varepsilon}$ for any $0 < \varepsilon < 1$. Hence we may always choose N sufficiently large so that (4.4) is satisfied. Clearly, if ε is very small, then N would need to be very large (cf. the comments in [10, Section 7]).

There is a problem in applying Theorem 4.1 in its present form, namely that in computing ϕ_N, (4.1) uses $\theta(0)$ which we cannot know precisely. This in general prevents an explicit calculation of some of the terms in (4.5). We therefore need to approximate $\{\phi_j\}_{j=0}^N$ in such a way that we replace (4.5) by terms which can be computed explicitly. In order to do this we assume henceforward that there exists some function $E(\lambda)$ such that

$$\begin{aligned}
&\left|\int_0^t f(s)\sin(2\lambda^{1/2}s)\,ds\right| \leqslant E(\lambda) \qquad \text{for all} \quad t\in[a,b]\\
&\left|\int_0^t f(s)\cos(2\lambda^{1/2}s)\,ds\right| \leqslant E(\lambda) \qquad \text{for all} \quad t\in[a,b]\\
&\qquad E(\lambda)\to 0 \qquad \text{as} \quad \lambda\to\infty
\end{aligned} \tag{4.7}$$

To illustrate how this may occur we cite

EXAMPLE 4.4. If $f(s)=s^{-\alpha}$ for some $\alpha<1$ and $E(\lambda)=C\lambda^{-\rho}$ where $\rho=\frac{1}{2}(1-\alpha)$ then (4.7) is satisfied for an appropriate constant C.

Analogous examples work if different values of α are chosen on $[a,0)$ and $(0,b]$ or if $f(s)$ is a finite sum of such powers of s, with a suitable choice of ρ in $E(\lambda)$. The proof of our claim in Example (4.4) is given at the end of Section 5 below. If, in the example, α is close to 1, then $E(\lambda)$ will be of larger order than $\lambda^{-1/2}$. However, by taking successive powers of $E(\lambda)$ we will eventually find some M for which $E(\lambda)^{M+1}=o(\lambda^{-1/2})$. It is for coefficients q in (1.1), which yield coefficients f which satisfy this type of property in (4.7), that we are able to provide a more useful, explicit version of Theorem 4.1.

Thus, we now assume that for $E(\lambda)$ in (4.7), there exists some integer power $M\geqslant 2$ for which

$$\begin{aligned}
&\lambda^{-1/2}<C(E(\lambda))^{M-1};\\
&C(E(\lambda))^{M}\leqslant\lambda^{-1/2},\\
&E(\lambda)^{M+1}=o(\lambda^{-1/2}) \qquad \text{as} \quad \lambda\to\infty.
\end{aligned} \tag{4.8}$$

We then define constants $\{\Psi_j\}$ and functions $\psi_{k,j}$ on $[a,b]$ by

$$\begin{aligned}
\Psi_1 &:= \theta(a)-\lambda^{1/2}a\\
\psi_{1,0}(t) &:= \Psi_1+\lambda^{1/2}t\\
\Psi_2 &:= \Psi_1-\int_a^0 f(s)\sin(\psi_{1,0}(s))\,ds\\
\psi_{k,0}(t) &:= \Psi_k+\lambda^{1/2}t\\
\psi_{k,j}(t) &:= \Psi_k+\lambda^{1/2}t-\int_0^t f(s)\sin(2\psi_{k,j-1}(s))\,ds\\
\Psi_{k+1} &:= \theta(a)-\lambda^{1/2}a-\int_a^0 f(s)\sin(2\psi_{k,N}(s))\,ds
\end{aligned} \tag{4.9}$$

for $k=2, ..., M$ and $j=1, ..., N$. This enables us to state our next main result for $\theta(b)-\theta(a)$.

THEOREM 4.5. *Assume that* (4.4), (4.7), *and* (4.8) *hold. Then as* $\lambda \to \infty$

$$\theta(b)-\theta(a)=\lambda^{1/2}(b-a)-\int_a^b f(t)\sin(2\psi_{M,N}(t))\,dt+\tfrac{1}{2}\lambda^{-1/2}\int_a^b F(t)\,dt$$

$$-\lambda^{-1/2}[f(t)(\sin^2\theta(t)-\sin^2(\psi_{1,0}(t)))]_a^b+o(\lambda^{-1/2}). \quad (4.10)$$

The proof which requires several preliminary lemmas and the use of Theorem 4.1 is given in Section 6 below.

5. PROOF OF THEOREM 4.1

We first need three preliminary lemmas which do not require (4.4) but in which we assume that $\lambda \geqslant 1$.

LEMMA 5.1. $|\phi_j(x)-\phi_{j-1}(x)| \leqslant C\xi_j(x)$ *for* $j=1, ..., N$ *and* $x \in [a, b]$.

Proof. We use induction on j. First for $j=1$, from (4.1),

$$\phi_1'(t)-\phi_0'(t)=-f(t)\sin(2\phi_0(t))+\lambda^{-1/2}F(t)\sin^2(\phi_0(t))$$

and $\phi_1(0)-\phi_0(0)=0$. So, integrating and using (4.2) we obtain

$$|\phi_1(x)-\phi_0(x)| \leqslant \left|\int_0^x |\phi_1'(t)-\phi_0'(t)|\,dt\right| \leqslant \left|\int_0^x |f(t)|+|F(t)|\,dt\right| = \xi_1(x)$$

as required for $j=1$. Now suppose the result true for j. Again using (4.1) we have

$$\phi_{j+1}'-\phi_j'(t)=-f(t)\{\sin(2\phi_j(t))-\sin(2\phi_{j-1}(t))\}$$
$$+\lambda^{-1/2}F(t)\{\sin^2\phi_j(t)-\sin^2\phi_{j-1}(t)\}$$

and $\phi_{j+1}(0)-\phi_j(0)=0$. Using the induction hypothesis and the Mean Value Theorem we have

$$\begin{aligned}|\phi_{j+1}'(t)-\phi_j'(t)| &\leqslant 2\,|f|\,|\sin(\phi_j-\phi_{j-1})|\,|\cos(\phi_j+\phi_{j-1})|+2\lambda^{-1/2}\,|F| \\ &\quad \times |\sin(\tfrac{1}{2}(\phi_j-\phi_{j-1}))|\,|\cos(\tfrac{1}{2}(\phi_j+\phi_{j-1}))|\,|\sin\phi_j+\sin\phi_{j-1}| \\ &\leqslant C(|f|+|F|)\,|\phi_j-\phi_{j-1}| \\ &\leqslant C(|f(t)|+|F(t)|)\,\xi_j(t).\end{aligned}$$

Here and elsewhere we suppress the dependent variable for the sake of brevity. Integration and (4.2) now yield

$$|\phi_{j+1}(x) - \phi_j(x)| \leqslant C \left| \int_0^x (|f(t)| + |F(t)|)\, \xi_j(t)\, dt \right| = \xi_{j+1}(x),$$

which suffices to complete the induction argument.

In the next two results we take θ to be the solution of (3.2).

LEMMA 5.2. $\theta(x) - \phi_{j-1}(x)| \leqslant C\xi_j(x)$ $(1 \leqslant j \leqslant N+1,\ x \in [a, b])$.

Proof. We again use induction on j and observe that (3.2) and (4.1) give that for $j = 1$,

$$\theta'(t) - \phi_0'(t) = -f \sin(2\theta) + \lambda^{-1/2} F \sin^2 \theta$$

and $\theta(0) - \phi_0(0) = 0$. Thus,

$$|\theta'(t) - \phi_0'(t)| \leqslant |f(t)| + |F(t)|$$

and hence from (4.2),

$$|\theta(x) - \phi_0(x)| \leqslant \xi_1(x).$$

Now supposing the result true for j and using (3.2) and (4.1),

$$\begin{aligned} \theta'(t) - \phi_j'(t) &= -f\{\sin(2\theta) - \sin(2\phi_{j-1})\} + \lambda^{-1/2} F\{\sin^2\theta - \sin^2 \phi_{j-1}\} \\ &= -2f \sin(\theta - \phi_{j-1}) \cos(\theta + \phi_{j-1}) + 2\lambda^{-1/2} F \\ &\quad \times \sin(\tfrac{1}{2}(\theta - \phi_{j-1})) \cos(\tfrac{1}{2}(\theta + \phi_{j-1}))(\sin\theta + \sin\phi_{j-1}). \end{aligned}$$

The induction hypothesis, Mean Value Theorem, and (4.10) then yield

$$\begin{aligned} |\theta'(t) - \phi_j'(t)| &\leqslant C(|f(t)| + |F(t)|)\, |\theta(t) - \phi_{j-1}(t)| \\ &\leqslant C(|f(t)| + |F(t)|)\, \xi_j(t). \end{aligned}$$

From this and (4.2) we obtain upon integration that

$$|\theta(x) - \phi_j(x)| \leqslant C\xi_{j+1}(x),$$

from which the result follows.

LEMMA 5.3. *For any $g \in L[a, b]$, as $\lambda \to \infty$,*

$$\text{(i)} \quad \int_a^b g(t) \sin \theta(t)\, dt = o(1);$$

$$\text{(ii)} \quad \int_a^b g(t) \sin \phi_j(t)\, dt = o(1), \qquad (0 \leqslant j \leqslant N).$$

Analogous results hold when sine is replaced by cosine.

Proof. (i) This is due to Atkinson [3]; the proof is reproduced in [10, Section 4].

(ii) This may be proved in the same was as (i), replacing θ by ϕ_j and using (4.1) in place of (3.2).

Proof of Theorem 4.1. An integration of (3.2) and use of trignometric identities gives

$$\theta(b) - \theta(a) = \lambda^{1/2}(b-a) - \int_a^b f(t) \sin(2\theta(t))\, dt + \tfrac{1}{2}\lambda^{-1/2} \int_a^b F(t)\, dt$$
$$- \tfrac{1}{2}\lambda^{-1/2} \int_a^b F(t) \cos(2\theta(t))\, dt. \tag{5.1}$$

Of the terms on the right hand side, the first and third occur in (4.5) and the fourth is $o(\lambda^{-1/2})$ by (2.7) and Lemma 5.3(i). So we need only consider the second term

$$\int_a^b f(t) \sin(2\theta(t))\, dt = \int_a^b f(t) \sin(2\phi_N(t))\, dt$$
$$+ \int_a^b f(t) \{\sin(2\theta(t)) - \sin(2\phi_N(t))\}\, dt \tag{5.2}$$

Since the first of the new terms on the right occurs in (4.5) we consider only the second term. We observe from (3.2) and (4.1) that

$$1 = \theta' \lambda^{-1/2} + \lambda^{-1/2} f \sin(2\theta) - \lambda^{-1} F \sin^2 \theta;$$
$$1 = \phi'_N \lambda^{-1/2} + \lambda^{-1/2} f \sin(2\phi_{N-1}) - \lambda^{-1} F \sin^2 \phi_{N-1}.$$

Thus we may rewrite the last term in (5.2) as

$$\int_a^b f \sin(2\theta) \{\theta' \lambda^{-1/2} + \lambda^{-1/2} f \sin(2\theta) - \lambda^{-1} F \sin^2 \theta\}\, dt$$
$$- \int_a^b f \sin(2\phi_N) \{\phi'_N \lambda^{-1/2} + \lambda^{-1/2} f \sin(2\phi_{N-1}) - \lambda^{-1} F \sin^2 \phi_{N-1}\}\, dt$$

$$= \lambda^{-1/2} \int_a^b f\{\sin(2\theta)\,\theta' - \sin(2\phi_N)\,\phi'_N\}\,dt$$

$$+ \lambda^{-1/2} \int_a^b f^2\{\sin^2(2\theta)) - \sin(2\phi_N)\sin(2\phi_{N-1})\}\,dt$$

$$- \lambda^{-1} \int_a^b fF\{\sin(2\theta)\sin^2\theta - \sin(2\phi_N)\sin^2\phi_{N-1}\}\,dt$$

$$=: I_1 + I_2 + I_3. \tag{5.3}$$

The existence of these integral terms is a consequence of assumption (4.4) and the steps below. We now examine I_1, I_2, and I_3 in turn.

Integration by parts gives

$$I_1 = \lambda^{-1/2}[f(t)(\sin^2(\theta(t)) - \sin^2(\phi_N(t))]_a^b$$
$$- \lambda^{-1/2} \int_a^b f'(t)(\sin^2(\theta(t)) - \sin^2(\phi_N(t))\,dt. \tag{5.4}$$

Lemma 5.2 with $j = N+1$, combined with (4.4), ensures that

$$|f(x)|\,|\theta(x) - \phi_N(x)| \leqslant C\,|f(x)\,\xi_{N+1}(x)| \to 0 \quad \text{as} \quad x \to 0 \tag{5.5}$$

and hence the boundary terms in (5.4) are bounded at $x = 0$. Indeed they vanish there. The boundary term of (5.4) occurs in (4.5) and the integral in (5.4) may be written as

$$-\lambda^{-1/2} \int_a^b f'(t)\{\sin\theta(t) - \sin\phi_N(t)\}\sin\theta(t)\,dt$$
$$-\lambda^{-1/2} \int_a^b f'(t)\{\sin\theta(t) - \sin\phi_N(t)\}\sin\phi_N(t)\,dt.$$

By Lemma 5.2, the Mean Value Theorem, and (4.4)

$$|f'(t)\{\sin\theta(t) - \sin\phi_N(t)\}| \leqslant C\,|f'(t)|\,|\theta(t) - \phi_N(t)|$$
$$\leqslant C\,|f'(t)|\,\xi_{N+1}(t) \in L[a, b].$$

Thus Lemma 5.3(i), (ii) yield that the last term in (5.4) is $o(\lambda^{-1/2})$.

In view of (5.1)–(5.4) it now suffices for the proof of Theorem 4.1 to prove that $I_2 + I_3 = o(\lambda^{-1/2})$. First we rewrite I_2 as

$$I_2 = \lambda^{-1/2} \int_a^b f^2\{\sin(2\theta) - \sin(2\phi_N)\} \sin(2\theta)\, dt$$
$$+ \lambda^{-1/2} \int_a^b f^2\{\sin(2\theta) - \sin(2\phi_N)\} \sin(2\phi_N)\, dt$$
$$+ \lambda^{-1/2} \int_a^b f^2\{\sin(2\phi_N) - \sin(2\phi_{N-1})\} \sin(2\phi_N)\, dt. \tag{5.6}$$

However, by the Mean Value Theorem and Lemmas 5.1 and 5.2

$$\begin{aligned} |\sin(2\theta(t)) - \sin(2\phi_N(t)| &\leqslant C\,|\theta(t) - \phi_N(t)| \leqslant C\xi_{N+1}(t); \\ |\sin(2\phi_N(t)) - \sin(2\phi_{N-1}(t))| &\leqslant C\,|\phi_N(t) - \phi_{N-1}(t)| \leqslant C\xi_N(t). \end{aligned} \tag{5.7}$$

It now follows from (4.4) and Lemma 5.3(ii) that each integral in (5.6) is $o(1)$ and hence $I_2 = o(\lambda^{-1/2})$.

Now, in (5.3) we write I_3 as

$$I_3 = \lambda^{-1} \int_a^b fF\{\sin(2\theta) - \sin(2\phi_N)\} \sin^2\theta\, dt$$
$$+ \lambda^{-1} \int_a^b fF(\sin\theta - \sin\phi_{N-1})(\sin\theta + \sin\phi_{N-1}) \sin(2\phi_N)\, dt.$$

In view of (5.7) and the analogous estimate that

$$|\sin(\theta(t)) - \sin(\phi_{N-1}(t))| \leqslant C\xi_N(t),$$

(4.4) and Lemma 5.3 now yield that $I_3 = o(\lambda^{-1/2})$ as required. We have also used the fact that, from (4.3), $\xi_{N+1}(t) \leqslant C\xi_N(t)$ on $[a, b]$. The proof of the theorem is now complete.

Proof of Corollary 4.2. The proof of Theorem 4.1 also holds if b is replaced by 0 throughout. The property (5.5) then ensures that the boundary term at 0 vanishes, as stated.

Proof of Example 4.4. Consider $\int_0^t s^{-\alpha} \sin(2\lambda^{1/2}s)\, ds$. If $0 \leqslant t \leqslant \lambda^{-1/2}$ then

$$\left| \int_0^t s^{-\alpha} \sin(2\lambda^{1/2}s)\, ds \right| \leqslant \int_0^t s^{-\alpha}\, ds = \frac{t^{1-\alpha}}{1-\alpha} \leqslant \frac{1}{1-\alpha} \lambda^{-(1/2)(1-\alpha)}.$$

If $t > \lambda^{-1/2}$ then letting $u := 2\lambda^{1/2}s$,

$$\int_0^t s^{-\alpha} \sin(2\lambda^{1/2}s)\, ds = \frac{1}{2\lambda^{1/2}} \int_0^{2\lambda^{1/2}t} \left(\frac{u}{2\lambda}\right)^{-\alpha} \sin(u)\, du$$
$$= \left(\frac{1}{2\lambda^{1/2}}\right)^{1-\alpha} \left\{ \int_0^\infty u^{-\alpha} \sin(u)\, du - \int_{2\lambda^{1/2}t}^\infty u^{-\alpha} \sin(u)\, du \right\}.$$

By the Second Mean Value Theorem, for any $X > 2\lambda^{1/2}t$, there is some $C(X)$ for which

$$\int_{2\lambda^{1/2}t}^{X} u^{-\alpha}\sin(u)\,du = (2\lambda^{1/2}t)^{-\alpha}\int_{C(X)}^{X}\sin(u)\,du$$

and hence

$$\left|\int_{2\lambda^{1/2}t}^{X} u^{-\alpha}\sin(u)\,du\right| \leqslant 2^{1-\alpha}.$$

Thus, using [13, p. 162] for the first infinite integral

$$\left|\int_0^t s^{-\alpha}\sin(2\lambda^{1/2}s)\,ds\right| \leqslant 2^{\alpha-1}\lambda^{-(1/2)(1-\alpha)}\left\{\Gamma(1-\alpha)\sin\left(\frac{\pi(1-\alpha)}{2}\right)+2^{1-\alpha}\right\}$$

as required. The proof is similar if cosine is used in place of sine throughout.

6. Proof of Theorem 4.5

The essence of Theorem 4.5 is the approximation of ϕ_N by the functions $\psi_{k,j}$ of (4.9). We explore this approximation process in the next six lemmas which lead to the proof of Theorem 4.5.

LEMMA 6.1. *For $j = 1, ..., N$ and $t \in [a, b]$ there exist functions $\varepsilon_j(t, \lambda)$ with*

$$\phi_j(t) - \psi_{1,0}(t) = \theta(0) - \Psi_1 + \varepsilon_j(t, \lambda)$$

and $|\varepsilon_j(t, \lambda)| \leqslant CE(\lambda)$.

Proof. We use induction on j. Consider first the case $j = 1$. By (4.1) and (4.9)

$$\phi_1(t) - \psi_{1,0}(t) = \theta(0) - \Psi_1 - \int_0^t f(s)\sin(2\theta(0) + 2\lambda^{1/2}s)\,ds$$

$$+\lambda^{-1/2}\int_0^t F(s)\sin^2(2\theta(0) + 2\lambda^{1/2}s)\,ds.$$

We use the addition formula for sine to expand the first integral as

$$\cos(2\theta(o))\int_0^t f(s)\sin(2\lambda^{1/2}s)\,ds + \sin(2\theta(0))\int_0^t f(s)\cos(2\lambda^{1/2}s)\,ds$$

and it follows that $\phi_1(t)-\psi_{1,0}(t)=\theta(0)-\Psi_1+\varepsilon_1(t,\lambda)$. Suppose now that the result is true for $j-1$; then, writing $\delta(t,\lambda)$ for a generic term which is $O(\lambda^{-1/2})$ as $\lambda\to\infty$, we have

$$\begin{aligned}\phi(t)-\psi_{1,0}(t)&=\theta(0)-\Psi_1-\int_0^t f(s)\sin(2\phi_{j-1}(s))\,ds+\delta(t,\lambda)\\&=\theta(0)-\Psi_1-\int_0^t f(s)\sin(2(\theta(0)-\Psi_1)+2\psi_{1,0}(s)\\&\quad+2\varepsilon_{j-1}(s,\lambda))\,ds+\delta(t,\lambda)\\&=\theta(0)-\Psi_1-\int_0^t f(s)\sin(2(\theta(0)-\Psi_1)+2\psi_{1,0}(s))\,ds\\&\quad-\int_0^t f(s)\{\sin(2(\theta(0)-\Psi_1)+2\psi_{1,0}(s)+2\varepsilon_{j-1}(s,\lambda))\\&\quad-\sin(2(\theta(0)-\Psi_1)+2\psi_{1,0}(s))\}\,ds+\delta(t,\lambda).\end{aligned}$$

By the addition formula for sine, the Mean Value Theorem, and the induction hypothesis, we deduce that

$$\begin{aligned}\phi_j(t)-\psi_{1,0}(t)&=\theta(0)-\Psi_1-\cos(2(\theta(0)-\Psi_1))\int_0^t f(s)\sin(2\psi_{1,1}(s))\,ds\\&\quad-\sin(2(\theta(0)-\Psi_1))\int_0^t f(s)\cos(2\psi_{1,1}(s))\,ds\\&\quad-2\int_0^t f(s)\sin(\varepsilon_{j-1}(s,\lambda))\cos(2(\theta(0)-\Psi_1))\\&\quad+2\psi_{1,0}(s)+\varepsilon_{j-1}(s,\lambda))\,ds.\end{aligned}\tag{6.1}$$

Using now the fact that $\psi_{1,0}(s)=\Psi_1+\lambda^{1/2}s$ we observe, as in the case $j=1$, that the first two integrals on the right of (6.1) are $O(E(\lambda))$. It follows from the induction hypothesis and the Mean Value Theorem that the last integral is

$$O(\max_{s\in[a,b]}|\varepsilon_{j-1}(s,\lambda)|)=O(E(\lambda)).$$

The proof is now complete.

LEMMA 6.2. $|\theta(0)-\Psi_1|\leqslant CE(\lambda)$.

Proof. By Corollary 4.2, as $\lambda\to\infty$

$$\theta(0)-\theta(a)=-\lambda^{1/2}a-\int_a^0 f(s)\sin(2\phi_N(s))\,ds+O(\lambda^{-1/2})$$

and by Lemma 6.1

$$\sin(2\phi_N(s)) = \cos(2(\theta(0) - \Psi_1)) \sin(2\psi_{1,0} + 2\varepsilon_N(s, \lambda)$$
$$+ \sin(2(\theta(0)) - \Psi_1) \cos(2\psi_{1,0} + 2\varepsilon_N(s, \lambda)).$$

Thus,

$$\begin{aligned}\int_a^0 f(s) \sin(2\phi_N(s))\, ds &= \cos(2(\theta(0) - \Psi_1)) \int_a^0 f(s) \sin(2\psi_{1,0}(s))\, ds \\ &\quad + \cos(2(\theta(0) - \Psi_1)) \int_a^0 f(s) \\ &\quad \times \{\sin(2\psi_{1,0}(s) + 2\varepsilon_N(s, \lambda)) - \sin(2\psi_{1,0}(s))\}\, ds \\ &\quad + \sin(2(\theta(0) - \Psi_1)) \int_a^0 f(s) \cos(2\psi_{1,0}(s))\, ds \\ &\quad + \sin(2(\theta(0) - \Psi_1)) \int_a^0 f(s) \\ &\quad \times \{\cos(2\psi_{1,0}(s) + 2\varepsilon_N(s, \lambda)) - \cos(2\psi_{1,0}(s))\}\, ds. \end{aligned} \tag{6.2}$$

The first and third terms on the right of (6.2) may be shown to be less than $CE(\lambda)$ as in the proof of Lemma 6.1, while the same bound may be obtained for the second and fourth terms by the Mean Value Theorem and Lemma 6.1.

LEMMA 6.3. *For $k = 2, ..., M$ and $t \in [a, b]$*

$$\text{(i)} \quad \left| \int_0^t f(s) \cos(2\psi_{k,j}(s))\, ds \right| \leqslant CE(\lambda)$$

$$\text{(ii)} \quad \left| \int_0^t f(s) \sin(2\psi_{k,j}(s))\, ds \right| \leqslant CE(\lambda)$$

for $j = 0, ..., N$.

Proof. We prove (i); the proof of (ii) is similar. We use induction on j. When $j = 0$

$$\int_0^t f(s) \cos(2\psi_{k,0}(s))\, ds = \int_0^t f(s) \cos(2\Psi_k + 2\lambda^{1/2}s)\, ds$$

and the result follows by the addition formula for cosine and (4.7).

Suppose now that the result is true for $j-1$. We write

$$\begin{aligned}\int_0^t f(s)\cos(2\psi_{k,j}(s))\,ds &= \int_0^t f(s)\cos\Big(2\Psi_k+2\lambda^{1/2}s\\&\qquad -2\int_0^s f(r)\sin(2\psi_{k,j-1}(r))\,dr\Big)\,ds\\&=\int_0^t f(s)\cos(2\Psi_k+2\lambda^{1/2}s)\,ds\\&\quad+2\int_0^t f(s)\sin\Big(\Big(\int_0^s f(r)\sin(2\psi_{k,j-1}(r)\,dr\Big)\\&\quad\times\sin\Big(2\Psi_k+2\lambda^{1/2}s-\int_0^s f(r)\sin(2\psi_{k,j-1}(r))\Big)\,dr\Big)\,ds\\&=: I_2+I_2.\end{aligned}$$

It is readily shown that $|I_1|\leqslant CE(\lambda)$ by an argument similar to the case $j=1$ and

$$\begin{aligned}|I_2|&\leqslant C\sup_{a\leqslant s\leqslant b}\left|\int_0^s f(r)\sin(2\psi_{k,j-1}(r))\,dr\right|\left|\int_0^b |f(s)|\,ds\right|\\&\leqslant CE(\lambda)\qquad\text{by the induction hypothesis.}\end{aligned}$$

LEMMA 6.4. *If for fixed k $|\theta(0)-\Psi_k|\leqslant CE(\lambda)$, but there is no $C>0$ with $C\,|\theta(0)-\Psi_k|\,E(\lambda)<\lambda^{-1/2}$ then for $j=0,\ldots,N$ there exists $\varepsilon_j(t,\lambda)$ with*

$$|\phi_j(t)-\psi_{k,j}(t)|=\theta(0)-\Psi_k+\varepsilon_j(t,\lambda),$$

where $|\varepsilon_j(t,\lambda)|\leqslant C\,|\theta(0)-\Psi_k|\,E(\lambda)$ for $t\in[a,b]$.

Proof. We use induction on j. Consider first the case $j=0$. From (4.1) and (4.9) we have

$$\phi_0(t)-\psi_{k,0}(t)=\theta(0)-\Psi_k$$

and the result is trivial. Suppose now that the result is true for $j-1$. We then have from (4.1) and (4.9) that

$$\begin{aligned}\phi_j(t)-\psi_{k,j}(t)&=\theta(0)-\Psi_k-\int_0^t f(s)\\&\qquad\times\{(2\phi_{j-1}(s))-\sin(2\psi_{k,j-1}(s))\}\,ds+\delta(t,\lambda),\end{aligned}$$

where $|\delta(t,\lambda)|\leqslant C\lambda^{-1/2}$. We then have by the addition formulae for sine and cosine, the induction hypothesis, and the Mean Value Theorem that

$$\phi_j(t) - \psi_{k,j}(t) = \theta(0) - \Psi_k + \delta(t,\lambda) - 2\sin(\theta(0) - \Psi_k)\int_0^t f(s)\cos(\varepsilon_{j-1}(s,\lambda))$$
$$\times\cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s) + \varepsilon_{j-1}(s,\lambda))\,ds$$
$$-2\cos(\theta(0) - \Psi_k)\int_0^t f(s)\sin(\varepsilon_{j-1}(s,\lambda))$$
$$\times\cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s) + \varepsilon_{j-1}(s,\lambda))\,ds$$

and hence that

$$|\phi_j(t) - \psi_{k,j}(t)| \leqslant \theta(0) - \Psi_k + \delta(t,\lambda) + I_1 + I_2, \tag{6.3}$$

where I_1 and I_2 denote the absolute value of the first and second integrals on the right hand side of (6.3), respectively. We note from the hypotheses that

$$|\delta(t,\lambda)| \leqslant C\lambda^{-1/2} \leqslant C\,|\theta(0) - \Psi_k|\,E(\lambda). \tag{6.4}$$

We further see easily that

$$I_2 \leqslant C \sup_{a \leqslant s \leqslant b} |\varepsilon_{j-1}(s,\lambda)| \left|\int_0^t |f(s)|\,ds\right| \leqslant C\,|\theta(0) - \Psi_k|\,E(\lambda)$$

by the induction hypothesis. To consider the term I_1, we observe first that

$$|\sin(\theta(0) - \Psi_k)| \leqslant C\,|\theta(0) - \Psi_k|. \tag{6.5}$$

Look now at the integral of I_1. This is bounded above by

$$\left|\int_0^t f(s)\cos(\theta(0)) - \Psi_k + 2\psi_{k,j-1}(s) + \varepsilon_{j-1}(s,\lambda))\,ds\right|$$
$$+\left|\int_0^t f(s)[1 - \cos(\varepsilon_{j-1}(s,\lambda))]\right.$$
$$\left.\times\cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s) + \varepsilon_{j-1}(s,\lambda))\,ds\right|$$
$$=: I_{11} + I_{12}.$$

It is readily seen by the induction hypothesis that

$$I_{12} \leqslant C \sup_{a \leqslant s \leqslant b} |1 - \cos(\varepsilon_{j-1}(s,\lambda))|$$
$$\leqslant C \sup_{a \leqslant s \leqslant b} |\varepsilon_{j-1}(s,\lambda)| \leqslant C\,|\theta(0) - \Psi_k|\,E(\lambda). \tag{6.6}$$

Also,

$$\begin{aligned}
I_{11} &\leqslant \left| \int_0^t f(s) \cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s))\, ds \right| \\
&\quad + \left| \int_0^t f(s) \{ \cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s) + \varepsilon_{j-1}(s)) \right. \\
&\quad \left. - \cos(\theta(0) - \Psi_k + 2\psi_{k,j-1}(s)) \}\, ds \right| \\
&=: I_{111} + I_{112}.
\end{aligned}$$

By Lemma 6.3

$$\begin{aligned}
I_{111} &\leqslant |\cos(\theta(0) - \Psi_k)| \left| \int_0^t f(s) \cos(2\psi_{k,j-1}(s)\, ds \right| \\
&\quad + |\sin(\theta(0) - \Psi_k)| \left| \int_0^t \sin(2\psi_{k,j-1}(s))\, ds \right| \\
&\leqslant CE(\lambda).
\end{aligned} \tag{6.7}$$

Finally, by the induction hypothesis and the Mean Value Theorem,

$$I_{112} \leqslant C \sup_{a \leqslant s \leqslant b} |\varepsilon_{j-1}(s, \lambda)| \leqslant CE(\lambda). \tag{6.8}$$

The result now follows from (6.3)–(6.8).

LEMMA 6.5. *If, for fixed k, $|\theta(0) - \Psi_k| \leqslant CE(\lambda)$ and there is no C with $|\theta(0) - \Psi_k|\, E(\lambda) < C\lambda^{-1/2}$ then*

$$|\theta(0) - \Psi_{k+1}| \leqslant C\, |\theta(0) - \Psi_k|\, E(\lambda).$$

Proof. We use Corollary 4.2 which states that

$$\theta(0) - \theta(a) = -\lambda^{1/2} a - \int_a^0 f(s) \sin(2\phi_N(s))\, ds + O(\lambda^{-1/2}).$$

We also have from Lemma 6.4 that

$$\phi_N(s) = \psi_{k,N}(s) + (\theta(0) - \Psi_k) + \varepsilon_N(s, \lambda),$$

where $|\varepsilon_N(s, \lambda)| \leqslant C\,|\theta(0) - \Psi_k|\, E(\lambda)$ for all $s \in [a, b]$. Thus,

$$\begin{aligned}\int_a^0 f(s) \sin(2\phi_N(s))\, ds = {} & \cos(2(\theta(0) - \Psi_k)) \int_a^0 f(s) \sin(2\psi_{k,N}(s))\, ds \\ & + \cos(2(\theta(0) - \Psi_k)) \int_a^0 f(s) \\ & \times \{\sin(2\psi_{k,N}(s) + 2\varepsilon_N(s, \lambda)) - \sin(2\psi_{k,N}(s))\} \\ & + \sin(2(\theta(0) - \Psi_k)) \int_a^0 f(s) \cos(2\psi_{k,N}(s))\, ds \\ & + \sin(2(\theta(0) - \Psi_k)) \int_a^0 f(s) \\ & \times \{\cos(2\psi_{k,N}(s) + 2\varepsilon_N(s, \lambda)) - \cos(2\psi_{k,N}(s))\}\, ds \\ =: {} & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{6.9}$$

We rewrite I_1 as

$$\begin{aligned} I_1 &= \int_a^0 f(s) \sin(2\psi_{k,N}(s))\, ds + (1 - \cos(2\theta(0) - \Psi_k)) \\ &\quad \times \int_a^0 f(s) \sin(2\psi_{k,N}(s))\, ds \\ &= -\Psi_{k+1} + \theta(a) - \lambda^{1/2} a + \delta(t, \lambda) \end{aligned}$$

by (4.8) where, by the Mean Value Theorem and Lemma 6.3,

$$|\delta(t, \lambda)| \leqslant C\,|\theta(0) - \Psi_k|\, E(\lambda).$$

It follows from the Mean Value Theorem that

$$|I_2| \leqslant C \sup_{a \leqslant s \leqslant b} |\varepsilon_N(s, \lambda)| \leqslant C\,|\theta(0) - \Psi_k|\, E(\lambda)$$

by Lemma 6.4. A similar argument establishes the same inequality for I_4. By the Mean Value Theorem and Lemma 6.3 we also have

$$I_3 \leqslant C\,|\theta(0) - \Psi_k|\, E(\lambda),$$

which completes the proof.

We examine now the case $k = M$.

LEMMA 6.6. *If there is no constant $C>0$ for which $|\theta(0)-\Psi_k|\leqslant C\lambda^{-1/2}$ but $|\theta(0)-\Psi_k|\ E(\lambda)\leqslant C\lambda^{-1/2}$ then*

$$\phi_j-\psi_{k,j}(t)=\theta(0)-\Psi_k+\varepsilon_j(t,\lambda)$$

for $j=0,\ldots,N$ and $t\in[a,b]$ where $|\varepsilon_j(t,\lambda)|\leqslant C\lambda^{-1/2}$.

Proof. We use induction on j. By (4.1) and (4.9) we have for the case $j=0$ that

$$\phi_0(t)-\psi_{k,0}=\theta(0)-\Psi_k \qquad \text{and} \qquad \varepsilon_0(t,\lambda)=0.$$

Suppose now that the result holds for $j-1$. We then have from (4.1) and (4.9) that

$$\phi_j(t)-\psi_{k,j}(t)=\theta(0)-\Psi_M-\int_0^t f(s)\{\sin(2\phi_{j-1}(s))-\sin(2\psi_{k,j-1}(s))\}\,ds$$
$$+\lambda^{-1/2}\int_0^t F(s)\sin^2(\phi_{j-1}(s))\,ds \tag{6.10}$$

By the induction hypothesis the first integral on the right hand side of (6.10) may be rewritten as

$$-2\int_0^t f(s)\sin(\phi_{j-1}(s)-\psi_{k,j-1}(s))\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds$$
$$=-2\int_0^t f(s)\sin(\theta(0)-\Psi_k+\varepsilon_{j-1}(s,\lambda))\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds$$
$$=-2\sin(\theta(0)-\Psi_k)\int_0^t f(s)\cos(\varepsilon_{j-1}(s,\lambda))\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds$$
$$-2\cos(\theta(0)-\Psi_k)\int_0^t f(s)\sin(\varepsilon_{j-1}(s,\lambda))\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds$$
$$=-2\sin(\theta(0)-\Psi_k)\int_0^t f(s)\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s,\lambda))\,ds$$
$$-2\sin(\theta(0)-\Psi_k)\int_0^t f(s)[1-\cos(\varepsilon_{j-1}(s,\lambda))]$$
$$\times\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds$$
$$-2\cos(\theta(0)-\Psi_k)\int_0^t f(s)\sin(\varepsilon_{j-1}(s,\lambda))\cos(\phi_{j-1}(s)+\psi_{k,j-1}(s))\,ds. \tag{6.11}$$

The terms of (6.11) may now be bounded in a similar way to the corresponding terms in the proof of Lemma 6.3.

We are now in a position to prove Theorem 4.5.

Proof of Theorem 4.5. We have from Theorem 4.1 that

$$\begin{aligned}\theta(b)-\theta(a) = \lambda^{1/2}(b-a) &- \int_a^b f(t)\sin(2\psi_{M,N}(t))\,dt + \tfrac{1}{2}\lambda^{-1/2}\int_a^b F(t)\,dt \\ &- \lambda^{-1/2}[f(t)(\sin^2\theta(t) - \sin^2\psi_{1,0}(t)]_a^b \\ &- \int_a^b f(t)[\sin(2\phi_N(t)) - \sin(2\psi_{M,N}(t))]\,dt \\ &- \lambda^{-1/2}f(t)(\sin^2\phi_N(t) - \sin^2\psi_{1,0}(t))_a^b + o(\lambda^{-1/2}) \qquad (6.12)\end{aligned}$$

as $\lambda\to\infty$. It is sufficient to show that the last two terms on the right hand side of (6.12) are $o(\lambda^{-1/2})$. This is easily shown for the last term by Lemma 6.1.

Consider now the integral term; this may be rewritten as

$$\begin{aligned}I := &-2\int_a^b f(t)\sin(\phi_N(t) - \psi_{M,N}(t))\cos(\phi_N(t) + \psi_{M,N}(t))\,dt \\ = &-2\int_a^b f(t)\sin(\theta(0) - \Psi_M + \varepsilon_N(t,\lambda))\cos(\phi_N(t) + \psi_{M,N}(t))\,dt \\ = &-2\sin(\theta(0) - \Psi_M)\int_a^b f(t)\cos(\varepsilon_N(t,\lambda))\cos(\phi_N(t) + \psi_{M,N}(t))\,dt \\ &-2\cos(\theta(0) - \Psi_M)\int_a^b f(t)\sin(\varepsilon_N(t,\lambda))\cos(\phi_N(t) + \psi_{M,N}(t))\,dt,\end{aligned}$$

where $|\varepsilon_N(t,\lambda)| \leqslant C\lambda^{-1/2}$ by Lemma 6.6. By (4.8) and Lemma 6.5 $|\theta(0) - \Psi_M| \leqslant CE(\lambda)^{M-1}$. Using Lemma 6.4 we have

$$\begin{aligned}I = &-2\sin(\theta(0) - \Psi_M)\int_a^b f(t)\cos(\phi_N(t) + \psi_{M,N}(t))\,dt \\ &-2\sin(\theta(0) - \Psi_M)\int_a^b f(t)[\cos(\varepsilon_N(t,\lambda)) - 1]\cos(\phi_N(t) + \psi_{M,N}(t))\,dt \\ &-\cos(\theta(0) - \Psi_M)\,\lambda^{-1/2}\int_a^b [f(t)\,\lambda^{1/2}\sin(\varepsilon_N(t,\lambda))\end{aligned}$$

$$\times \cos(\psi_{M,N}(t))] \cos(\phi_N(t))\, dt$$

$$+\cos(\theta(0)-\Psi_M)\, \lambda^{-1/2} \int_a^b [f(t)\, \lambda^{1/2} \sin(\varepsilon_N(t, \lambda))$$

$$\times \sin(\psi_{M,N}(t))] \sin(\phi_N(t))\, dt$$

$$=: I_1 + I_2 + I_3 + I_4. \tag{6.13}$$

I_3 and I_4 are $o(\lambda^{-1/2})$ as $\lambda \to \infty$ by Lemma 5.3(ii) since $f(t)\, \lambda^{1/2}$ $\sin(\varepsilon_N(t, \lambda)) \cos(\psi_{M,N}(t))$ and $f(t)\, \lambda^{1/2} \sin(\varepsilon_N(t, \lambda)) \sin(\psi_{M,N}(t))$ both belong to $L^1[a, b]$ uniformly in λ.

By Lemma 6.6

$$|I_2| \leqslant C\, |\theta(0)-\Psi_M| \sup_{a \leqslant s \leqslant b} |\cos(\varepsilon_N(s, \lambda))-1|$$

$$\leqslant CE(\lambda)^{M-1}\, \lambda^{-1/2} = o(\lambda^{-1/2}).$$

Consider now the integral of I_1. By Lemma 6.4 this may be rewritten as

$$\int_a^b f(t) \cos(\theta(0)-\Psi_M + 2\psi_{M,N}(t) + \varepsilon_N(t, \lambda))\, dt$$

$$= \cos(\theta(0)-\Psi_M) \int_a^b f(t) \cos(2\psi_{M,N}(t) + \varepsilon_N(t, \lambda))\, dt$$

$$- \sin(\theta(0)-\Psi_M) \int_a^b f(t) \sin(2\psi_{M,N}(t) + \varepsilon_N(t, \lambda))\, dt$$

$$=: I_{11} + I_{12}.$$

Now, by Lemma 6.5 and the Mean Value Theorem

$$|I_1| \leqslant CE(\lambda)^{M-1} (|I_{11}| + |I_{12}|). \tag{6.14}$$

We write

$$I_{11} = \cos(\theta(0)-\Psi_M) \int_a^b f(t) \cos(2\psi_{M,N}(t))\, dt + \cos(\theta(0)-\Psi_M)$$

$$\times \int_a^b f(t)[\cos(2\psi_{M,N}(t) + \varepsilon_N(t, \lambda) - \cos(2\psi_{M,N}(t))]\, dt$$

$$= I_{111} + I_{112}.$$

By Lemma 6.3 and (4.8)

$$|I_{111}| \leqslant CE(\lambda)$$

and by Lemma 6.4 and the Mean Value Theorem

$$|I_{112}| \leqslant C \sup_{a \leqslant s \leqslant b} |\varepsilon_N(s, \lambda)| \leqslant CE(\lambda).$$

Turning now to I_{12} we have by Lemma 6.5 and the Mean Value Theorem that

$$|I_{12}| \leqslant |\sin(\theta(0) - \Psi_M)| \leqslant |\theta(0) - \Psi_M| \leqslant CE(\lambda)^{M-1}.$$

The result now follows from (6.13) and (6.14).

7. An Example

To illustrate the foregoing results we consider the example

$$-y''(x) + q(x)\, y(x) = \lambda y(x) \qquad \text{for} \quad -1 \leqslant x \leqslant 1, \tag{7.1}$$

where

$$q(x) := \begin{cases} x^{-3/2} & 0 < x < 1 \\ -(-x)^{-3/2} & -1 < x < 0 \end{cases}$$

with Dirichlet boundary conditions at -1 and 1. This example is beyond the range of the results of [3] and [10]. We use the regularization of [10, Section 7] and set

$$f(x) = 2\,|x|^{-1/2} + 4 \log |x| \qquad \text{for} \quad x \in [-1, 1]/\{0\}; \tag{7.2}$$

with this choice of f we have from (2.5) that

$$F(x) = 16\,|x|^{-1/2} \log |x| + 16(\log |x|)^2. \tag{7.3}$$

We observe that f and F are both even functions of x. In the notation of (4.2) we have, as $x \to 0$,

$$\xi_1(x) = \int_0^x |f(t)| + |F(t)|\, dt = O(|x|^{-1/2})$$

$$\xi_2(x) = O(|x|) \qquad f'(x) = O(|x|^{-3/2}) \qquad f'(x)\, \xi_2(x) = O(|x|^{-1/2})$$

$$f(x)^2 = O(|x|^{-1}) \qquad f(x)^2\, \xi_1(x) = O(|x|^{-1/2}) \qquad f(x)\, F(x) = O(|x|^{-1})$$

$$f(x)\, F(x)\, \xi_1(x) = O(|x|^{-1/2}) \qquad \text{and} \qquad f\xi_2 \text{ is continuous at } x = 0.$$

We may thus take $N = 1$ in Theorem 4.5. It may also be seen as in Example 4.4 that $E(\lambda) = c\lambda^{-1/4}$ so in (4.8) we may take $M = 2$. The relevant functions from (4.9) in this case are

$$\Psi_1 = \lambda^{1/2}$$

$$\psi_{1,0} = \lambda^{1/2} + \lambda^{1/2}t$$

$$\Psi_2 = \Psi_1 - \int_{-1}^{0} f(s)\sin(2\lambda^{1/2}s + 2\lambda^{1/2})\,ds$$

$$\psi_{2,0} = \Psi_2 + \lambda^{1/2}t$$

$$\psi_{2,1} = \Psi_2 + \lambda^{1/2}t - \int_0^t f(s)\sin(2\Psi_2 + 2\lambda^{1/2}s)\,ds.$$

The main task is to determine the asymptotic form of these functions to within a $o(\lambda^{-1/2})$ term. It may readily be shown that

$$\psi_{2,1}(t) = \Psi_2 + \lambda^{1/2}t - \delta(t,\lambda), \tag{7.4}$$

where

$$\delta(t,\lambda) = O(\lambda^{-1/4}) \qquad \text{as} \quad \lambda \to \infty$$

and

$$\delta(t,\lambda) = \delta_1(t,\lambda) + \delta_2(t,\lambda),$$

where

$$\begin{aligned} \delta_1(t,\lambda) &:= \int_0^t f(s)\cos(2\lambda^{1/2}s)\,ds && \text{is an odd function} \\ \delta_2(t,\lambda) &:= \int_0^t f(s)\sin(2\lambda^{1/2}s)\,ds && \text{is an even function.} \end{aligned} \tag{7.5}$$

In order to apply Theorem 4.5 to the example we need to evaluate $\int_{-1}^{1} f(t)\sin(2\psi_{2,1}(t))\,dt$ to within a $o(\lambda^{-1/2})$ term.

It is helpful now to gather together for future use some asymptotic results which are required in the sequel.

$$\begin{aligned} \int_0^\infty t^{\rho-1}\cos(t)\,dt &= \Gamma(\rho)\cos\left(\frac{\rho\pi}{2}\right) \qquad \text{for} \quad \rho > 0 \\ \int_0^\infty t^{\rho-1}\sin(t)\,dt &= \Gamma(\rho)\sin\left(\frac{\rho\pi}{2}\right). \end{aligned} \tag{7.6}$$

In particular

$$\int_0^\infty t^{-1}\sin t\,dt = \frac{\pi}{2}. \tag{7.7}$$

The following may be shown by standard techniques:

$$\begin{aligned} \int_0^1 t^{-1/2}\cos(2\lambda^{1/2}t)\,dt &= \frac{\pi^{1/2}}{2}\lambda^{-1/4} + \frac{1}{2}\sin(2\lambda^{1/2})\,\lambda^{-1/2} + o(\lambda^{-1/2}) \\ \int_0^1 t^{-1/2}\sin(2\lambda^{1/2}t)\,dt &= \frac{\pi^{1/2}}{2}\lambda^{-1/4} - \frac{1}{2}\cos(2\lambda^{1/2})\,\lambda^{-1/2} + o(\lambda^{-1/2}) \end{aligned} \tag{7.8}$$

$$\begin{aligned} \int_0^1 (\log t)\cos(2\lambda^{1/2}t)\,dt &= -\frac{\pi}{4}\lambda^{-1/2} + o(\lambda^{-1/2}) \\ \int_0^1 (\log t)\sin(2\lambda^{1/2}t)\,dt &= \left\{\frac{1}{2}\int_0^1 u^{-1}(\cos u - 1)\,du + \frac{1}{2}\int_1^\infty u^{-1}\cos u \right. \\ &\quad \left. - \frac{1}{2}\log(2\lambda^{1/2})\right\}\lambda^{-1/2} + o(\lambda^{-1/2}). \end{aligned} \tag{7.9}$$

In consequence of (7.8) and (7.9)

$$\begin{aligned} \int_{-1}^0 (-s)^{-1/2}\cos(2\lambda^{1/2}s)\,ds &= \frac{\pi^{1/2}}{2}\lambda^{-1/4} + O(\lambda^{-1/2}) \\ \int_{-1}^0 (-s)^{-1/2}\sin(2\lambda^{1/2}s)\,ds &= -\frac{\pi^{1/2}}{2}\lambda^{-1/4} + O(\lambda^{-1/2}) \end{aligned} \tag{7.10}$$

$$\begin{aligned} \int_{-1}^0 \log(-s)\cos(2\lambda^{1/2}s)\,ds &= O(\lambda^{-1/2}) \\ \int_{-1}^0 \log(-s)\sin(2\lambda^{1/2}s)\,ds &= O(\lambda^{-1/2}\log\lambda). \end{aligned} \tag{7.11}$$

We define the constants

$$\begin{aligned} A &:= \tfrac{1}{2}\int_0^\infty v^{-1/2}\sin v \int_0^v u^{-1/2}\cos u\,du\,dv \\ B &:= \int_0^1 (\log t)^2\,dt \\ E &:= \tfrac{1}{2}\int_0^1 t^{-1/2}\log t\,dt. \end{aligned}$$

It may further be shown that

$$\int_0^1 t^{-1/2}\sin(2\lambda^{1/2}t)\int_0^t s^{-1/2}\cos(2\lambda^{1/2}s)\,ds\,dt = A\lambda^{-1/2} + o(\lambda^{-1/2}) \tag{7.12}$$

$$2\int_0^1 t^{-1/2}\sin(2\lambda^{1/2}t)\int_0^t \log s\cos(2\lambda^{1/2}s)\,ds\,dt = E\lambda^{-1/2} + o(\lambda^{-1/2}) \tag{7.13}$$

$$2\int_0^1 \log t \sin(2\lambda^{1/2}t)\int_0^t s^{-1/2}\cos(2\lambda^{1/2}s)\,ds\,dt = E\lambda^{-1/2}+o(\lambda^{-1/2})$$

$$4\int_0^1 \log t \sin(2\lambda^{1/2}t)\int_0^t \log s\cos(2\lambda^{1/2}s)\,ds\,dt = B\lambda^{-1/2}+o(\lambda^{-1/2}). \quad (7.14)$$

We first establish the asymptotic form of Ψ_2. We have that

$$\begin{aligned}\Psi_2 &= \lambda^{1/2} - \int_{-1}^{0} [2(-s)^{-1/2} + 4\log(-s)]\sin(2\lambda^{1/2}s + 2\lambda^{1/2})\,ds \\ &= \lambda^{1/2} - \pi^{1/2}\lambda^{-1/4}(\cos(2\lambda^{1/2}) - \sin(2\lambda^{1/2})) + O(\lambda^{-1/2}\log\lambda) \quad (7.15)\end{aligned}$$

by (7.10) and (7.11). It follows from (7.15) that

$$\begin{aligned}\sin(2\Psi_2) = {}&\sin(2\lambda^{1/2})[1 + 2\pi^{1/2}\lambda^{-1/4}\cos(2\lambda^{1/2})] \\ &- 2\pi^{1/2}\lambda^{-1/4}\cos^2(2\lambda^{1/2}) + O(\lambda^{-1/2}\log\lambda) \quad (7.16)\end{aligned}$$

and

$$\begin{aligned}\cos(2\Psi_2) = {}&\cos(2\lambda^{1/2})[1 + 2\pi^{1/2}\lambda^{-1/4}\cos(2\lambda^{1/2})] \\ &- 2\pi^{1/2}\lambda^{-1/4}\sin^2(2\lambda^{1/2}) + O(\lambda^{-1/2}\log\lambda). \quad (7.17)\end{aligned}$$

Passing now to $\int_{-1}^{1} f(t)\sin(2\psi_{2,1}(t))\,dt$ we have from (7.4) and (7.5) that

$$\begin{aligned}&\int_{-1}^{1} f(t)\sin(2\psi_{2,1}(t))\,dt \\ &= \int_{-1}^{1} f(t)\sin(2\Psi_2 + 2\lambda^{1/2}t - 2\delta(t,\lambda))\,dt \\ &= \cos(2\Psi_2)\int_{-1}^{1} f(t)\sin(2\lambda^{1/2}t)\,dt \\ &\quad + \cos(2\Psi_2)\int_{-1}^{1} f(t)[\sin(2\lambda^{1/2}t - 2\delta(t,\lambda)) - \sin(2\lambda^{1/2}t)]\,dt \\ &\quad + \sin(2\Psi_2)\int_{-1}^{1} f(t)\cos(2\lambda^{1/2}t)\,dt \\ &\quad + \sin(2\Psi_2)\int_{-1}^{1} f(t)[\cos(2\lambda^{1/2}t - 2\delta(t,\lambda)) - \cos(2\lambda^{1/2}t)]\,dt.\end{aligned}$$

Since f is even, we then have

$$\int_{-1}^{1} f(t)\sin(2\psi_{2,1}(t))\,dt$$

$$= 2\sin(2\Psi_2)\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt$$

$$+ 2\cos(2\Psi_2)\int_{-1}^{1} f(t)\sin(-\delta(t,\lambda))\cos(2\lambda^{1/2}t-\delta(t,\lambda))\,dt$$

$$- 2\sin(2\Psi_2)\int_{-1}^{1} f(t)\sin(-\delta(t,\lambda))\sin(2\lambda^{1/2}t-\delta(t,\lambda))\,dt$$

$$= 2\sin(2\Psi_2)\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt$$

$$- 2\cos(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)\cos(2\lambda^{1/2}t-\delta(t,\lambda))\,dt$$

$$+ 2\sin(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)\sin(2\lambda^{1/2}t)-\delta(t,\lambda))\,dt + O(\lambda^{-3/4})$$

$$= 2\sin(2\Psi_2)\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt$$

$$+ 2\sin(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)\sin(2\lambda^{1/2}t)$$

$$- 2\cos(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)\cos(2\lambda^{1/2}t)\,dt$$

$$- 2\cos(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)[\cos(2\lambda^{1/2}t-\delta(t,\lambda))-\cos(2\lambda^{1/2}t)]\,dt$$

$$+ 2\sin(2\Psi_2)\int_{-1}^{1} f(t)\,\delta(t,\lambda)[\sin(2\lambda^{1/2}t-\delta(t,\lambda))-\sin(2\lambda^{1/2}t)]\,dt$$

$$+ O(\lambda^{-3/4})$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5 + O(\lambda^{-3/4}). \tag{7.18}$$

We deal separately with the integrals of (7.18).

Consider first I_5. This may be rewritten as

$$-4\lambda^{-1/2}\sin(2\Psi_2)\int_{-1}^{1}\{f(t)\,\lambda^{1/2}\,\delta(t,\lambda)\sin(\delta(t,\lambda))\}\cos(2\lambda^{1/2}t)\,dt$$

$$-4\sin(2\Psi_2)\int_{-1}^{1}\delta(t,\lambda)\,f(t)\sin(\delta(t,\lambda))$$
$$\times\{\cos(2\lambda^{1/2}t-\delta(t,\lambda))-\cos(2\lambda^{1/2}t)\}\,dt.$$

The first term is $o(\lambda^{-1/2})$ by the Riemann Lebesgue Lemma since the factor $\{\cdots\}$ in the integrand is in $L[-1,1]$ uniformly in λ by (7.4). The second term is $O(\lambda^{-3/4})$ by (7.4) and the Mean Value Theorem. A similar analysis applies to I_4.

Consider now I_2. By (7.5)

$$I_2=4\sin(2\Psi_2)\int_0^1 f(t)\left(\int_0^t f(s)\cos(2\lambda^{1/2}s)\,ds\right)\sin(2\lambda^{1/2}t)\,dt$$

$$=4\sin(2\Psi_2)\int_0^1(2t^{-1/2}+4\log t)$$
$$\times\left(\int_0^t(2s^{-1/2}+4\log s)\cos(2\lambda^{1/2}s)\,ds\right)\sin(2\lambda^{1/2}t)\,dt$$

$$=16\sin(2\Psi_2)\left\{\int_0^1 t^{-1/2}\sin(2\lambda^{1/2}t)\int_0^t s^{-1/2}\cos(2\lambda^{1/2}s)\,ds\,dt\right.$$
$$+2\int_0^1 t^{-1/2}\sin(2\lambda^{1/2}t)\int_0^t\log s\cos(2\lambda^{1/2}s)\,ds\,dt$$
$$+2\int_0^1\log t\sin(2\lambda^{1/2}t)\int_0^t s^{-1/2}\cos(2\lambda^{1/2}s)\,ds\,dt$$
$$\left.+4\int_0^1\log t\sin(2\lambda^{1/2}t)\int_0^t\log s\cos(2\lambda^{1/2}s)\,ds\,dt\right\}$$

$$=16\sin(2\Psi_2)\{A+2E+B\}\,\lambda^{-1/2}+o(\lambda^{-1/2})\qquad\text{by (7.12)–(7.14)}.$$

Turning now to I_3 we have by (7.5) that

$$I_3=-4\cos(2\Psi_2)\int_0^1 f(t)\int_0^t f(s)\sin(2\lambda^{1/2}s)\,ds\cos(2\lambda^{1/2}t)\,dt$$

$$=-4\cos(2\Psi_2)\left\{\left(\int_0^1 f(t)\sin(2\lambda^{1/2}t)\,dt\right)\left(\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt\right)\right.$$
$$\left.-\int_0^1 f(s)\sin(2\lambda^{1/2}s)\int_0^s f(t)\cos(2\lambda^{1/2}t)\,dt\,ds\right\}.$$

By (7.8) and (7.9)

$$\int_0^1 f(t)\sin(2\lambda^{1/2}t)\,dt = \pi^{1/2}\lambda^{-1/4} + O(\lambda^{-1/2}\log\lambda)$$

and

$$\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt = \pi^{1/2}\lambda^{-1/4} + O(\lambda^{-1/2}\log\lambda)$$

and we have shown that

$$\int_0^1 f(s)\sin(2\lambda^{1/2}s)\int_0^s f(t)\cos(2\lambda^{1/2}t)\,dt\,ds = (A+B+2E)\,\lambda^{-1/2} + o(\lambda^{-1/2})$$

so $I_3 = -4\cos(2\Psi_2)\{\pi - (A+B+2E\}\,\lambda^{-1/2} + o(\lambda^{-1/2})$. Finally,

$$\begin{aligned} I_1 &= 2\sin(2\Psi_2)\int_0^1 f(t)\cos(2\lambda^{1/2}t)\,dt \\ &= 2\sin(2\Psi_2)(\pi^{1/2}\lambda^{-1/4} + [\sin(2\lambda^{1/2}) - \pi]\,\lambda^{-1/2}) + o(\lambda^{-1/2}). \end{aligned}$$

We thus have by (7.16) and (7.17) that

$$\begin{aligned} &\int_{-1}^1 f(t)\sin(2\psi_{2,1}(t))\,dt \\ &\quad = 2\pi^{1/2}\sin(2\lambda^{1/2})\,\lambda^{-1/4} + \lambda^{-1/2}[\sin(2\lambda^{1/2}) \\ &\qquad \times\{4\pi\cos(2\lambda^{1/2}) + 2\sin(2\lambda^{1/2}) - \pi + 16(A+B+2E)\} \\ &\qquad - 4\cos(2\lambda^{1/2})\{\pi\cos(2\lambda^{1/2}) + \pi - A - B - 2E\}] + o(\lambda^{-1/2}). \end{aligned} \tag{7.19}$$

In order to apply Theorem 4.5 we also need to know that

$$\begin{aligned} \tfrac{1}{2}\lambda^{-1/2}\int_{-1}^1 F(t)\,dt &= \lambda^{-1/2}\int_0^1 16t^{-1/2}\log t + 16(\log t)^2\,dt \\ &= \lambda^{-1/2}(32E + 16B). \end{aligned} \tag{7.20}$$

The Dirichlet conditions at -1 and 1 imply that $\theta(-1) = 0$ and $\theta(1) = (n+1)\pi$ so

$$\begin{aligned} &-\lambda^{-1/2}[f(t)(\sin^2\theta(t) - \sin^2\psi_{1,0}(t))]|_{-1}^{1} \\ &\quad = \lambda^{-1/2}(2\,|t|^{-1/2} + 4\log|t|)\sin^2\psi_{1,0}(t)|_{-1}^{1} \\ &\quad = 2\lambda^{-1/2}\sin^2(2\lambda^{1/2}). \end{aligned} \tag{7.21}$$

We now have from Theorem 4.5 and (7.19), (7.20), (7.21) that

$$
\begin{aligned}
(n+1)\pi = {} & 2\lambda_n^{1/2} + 2\pi^{1/2}\sin(2\lambda_n^{1/2})\,\lambda^{-1/4} + \lambda_n^{-1/2}[\sin(2\lambda_n^{1/2}) \\
& \times \{4\pi\cos(2\lambda_n^{1/2}) + 4\sin(2\lambda_n^{1/2}) - \pi + 16A - 4\cos(2\lambda_n^{1/2}) \\
& \times \{\pi\cos(2\lambda_n^{1/2}) + \pi - A - B - 2E\} + 32E + 16B] + o(\lambda_n^{-1/2}) \\
& \text{as} \quad n \to \infty.
\end{aligned}
$$

Reversion now yields

$$
\begin{aligned}
\lambda_n = {} & \frac{(n+1)\pi}{2} - \frac{2^{1/2}\delta_n}{(n+1)^{1/2}} - \frac{1}{(n+1)\pi}\{\delta_n(4\delta_n - \pi + 16A) \\
& + 16B(\delta_n + 1) + 16E(\delta_n + 2)\} + o(n^{-1}) \qquad \text{as} \quad n \to \infty,
\end{aligned}
$$

where

$$
\delta_n = \begin{cases} 1 & \text{if} \quad n \equiv 0 \bmod 4 \\ -1 & \text{if} \quad n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}
$$

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